Lecture 7: Eigenvalues and Diagonalization

Linear Algebra II · Autumn term 2026

Department of Mathematics

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1 Eigenvalues and eigenvectors

Let A be an $n \times n$ matrix. A non-zero vector $\mathbf{v}$ is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{1}$$

Rewriting Eq. (1) as $(A - \lambda I)\mathbf{v} = \mathbf{0}$ shows that a non-zero solution exists exactly when $A - \lambda I$ is singular, that is, when

$$\det(A - \lambda I) = 0. \tag{2}$$

The left-hand side of Eq. (2) is a polynomial of degree n in λ, the **characteristic polynomial** of A.

> **Example 7.1**
>
> For $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$, $\det(A - \lambda I) = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$, so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$.

2 Diagonalization

A matrix A is **diagonalizable** if there is an invertible matrix P and a diagonal matrix D with

$$A = PDP^{-1}, \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}. \tag{3}$$

> **Note**
>
> **Theorem 7.2.** An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors. In that case the columns of P in Eq. (3) are those eigenvectors.

Diagonalization makes powers easy to compute: $A^k = PD^kP^{-1}$, and D^k only needs the k-th powers of the eigenvalues.[1]

3 Summary

Table 1 Facts to remember

Property	Condition
λ is an eigenvalue	$\det(A - \lambda I) = 0$
A is invertible	0 is not an eigenvalue
A is diagonalizable	n linearly independent eigenvectors
$\operatorname{tr} A$	$\lambda_1 + \lambda_2 + \cdots + \lambda_n$
$\det A$	$\lambda_1 \lambda_2 \cdots \lambda_n$

4 Exercises

1. Find the eigenvalues and eigenvectors of $\left(\begin{smallmatrix} 2 & 0 \\ 1 & 3 \end{smallmatrix}\right)$.
2. Show that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not diagonalizable.
3. Use diagonalization to compute A^{10} for the matrix of Example 7.1.

[1] This is how the closed form of the Fibonacci numbers is usually derived.